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FRACTAL GEOMETRY: MATHEMATICAL TOOL FOR MODELLING THE NATURAL WORLD

by

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1.1 Introduction

Mathematics has developed into what is arguably the richest and most creative activity of the human mind that has developed broadly in scope and branches. Many of us left School thinking that Mathematics is a cut and dried subject constructed many years ago by a cold, logical process to be studied subsequently by generations of unnatural students whereas, the real stuff of the subject is found in something familiar to us; *Numbers and Shapes*

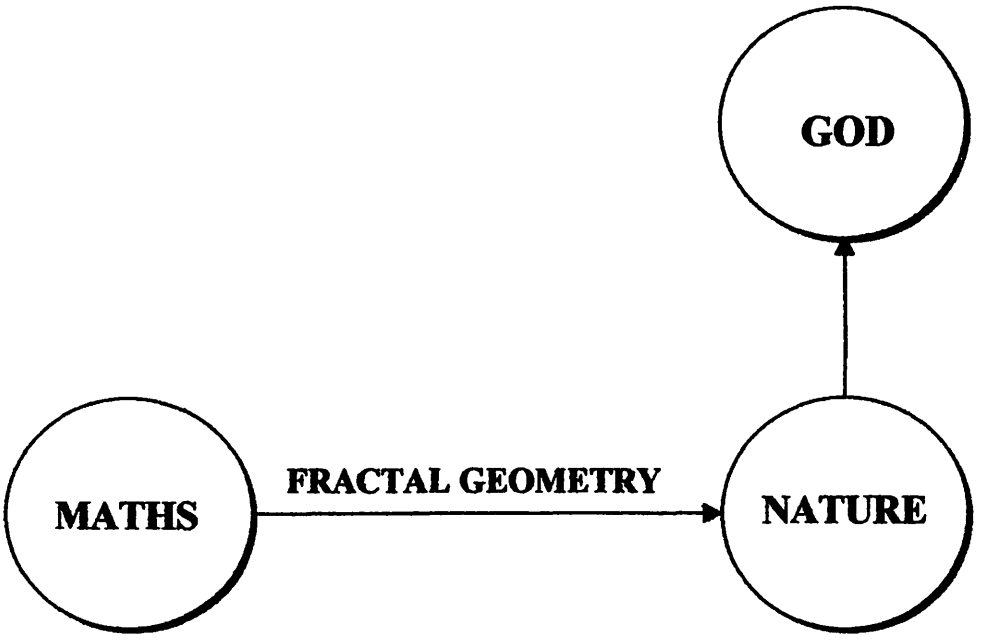
Today, on the ground of both its practical value and its appeal to the intellect, its contribution in the development strategies have been very significant. However, due to its nature and its daily effects many questions about the subject have come to mind; what is Mathematics? How did it originate? What are its concerns? How does it fit within the varieties of human experiences? These questions are not made easier by the amount of material required and the extensive interlinking which in effect makes it impossible for us to answer them satisfactorily as should be. However, I shall restrict myself to a new focus of mathematical research called 'fractal geometry' a concept developed by Mandelbrot [1] which has found much usefulness in the natural world to answer these questions though not exhaustively.

The natural world is filled with intricate details such as geometrical properties of such objects such as the shore of continents, the branches of trees or the surface of clouds, the intricate structure of a black spleenwort

fern, the geometry on the back of your hand: the fine pores, the fine lines and the color variation, the coastlines, the distribution of sizes of islands and lakes, the discharges of rivers, the variations of climates, mountains, plants, insects and cells, rains and cloud areas, as well as the geometrical structure of crystals. All these can be interestingly described, modeled and analyzed by fractal geometry. God has created many interesting and beautiful things in nature which can serve as an expression of the intelligence of the Designer of the universe. Therefore, man needs to study nature in order to appreciate nature and its designer rather than using philosophy to determine how nature must behave. In contrast, the Greeks modeled nature indirectly using philosophy, rather than directly from nature itself.

Our main objective in this paper is to establish the fact that Mathematics has evolved precisely as a symbolic representation of the universe and that through the study of mathematics, in particular fractal geometry, we can learn of beautiful structures and shapes that can create in us love ,respect and allegiance for the designer of the universe

The diagram in fig. 1 shows a schematic representation of the aim of this paper.



The structure of this paper is as follows. In section 1.2, among many different definitions, I choose to define mathematics as the science of quantity and space while the emphasis is on the aspect of space in which geometric objects are described . Abstraction is one of the attributes of mathematics that plays important roles in the hands of Mathematicians, some of these roles are briefly discussed in relationship with physical problems.The excellent demonstration of pattern and symmetry by some natural objects and phenomena give credence to their creator. I try to establish the fact that God is a Mathematician since some of the natural objects in nature can be interpreted using mathematics.

In section 1.3, I mention some truths in mathematics as a means of integrating faith in God with learning mathematics. During the last two decades it has widely been recognized by Physicists working in diverse areas that many of the structures common in their experiments possess a rather

special kind of geometrical complexity. This awareness is largely due to the activity of Mandelbrot [1,2] who called particular attention to the geometrical properties of some natural objects and phenomena such as mentioned previously. He coined the name 'fractal' for those complex shapes to express that they can be characterized by a *non-integer dimensionality.* Thus, in section 2.1, I discuss briefly this concept of fractal geometry as a modeling tool different from the Euclidean geometry, also the characteristics of a fractal set are discussed in section 2.3.In section 2.3,some basic truths in this concept are discussed to illustrate integration of faith with the study of fractals.

In section 2.4 the fundamental approach to fractal geometry through 'iterated function systems' are discussed and some applications of fractals geometry are considered in section 2.5.

The conclusion of the paper is in section 2.6.

1.2 **Mathematics and The Design of The Universe**

As Christian Teachers, it is essential to acknowledge the fact that "all truth is God's truth and every field of study including Mathematics can broaden and deepen our understanding of truth as revealed in Jesus, the Bible and Nature"[4]

Isaac Watts declares,

"Nature with open volume stands,

To spread its Maker's praise abroad;

And every labor of His hands

Shows something worthy of our God"[5]

Thus, nature as an open book can be studied in the mathematics class using fractal geometry.

Many authors have defined mathematics in different ways but I have chosen for the purpose of this paper , to define mathematics as the science of *quantity* and *space*. However, this definition has been modified and extended by many authors in a way that reflects the growth of the subject over the past several centuries.

The science of quantity and space in their simple forms are known as *arithmetic* and *geometry*. One of the attributes of mathematics that scares people is its abstraction. Abstraction is the process of learning to work with objects rather than specific items. For instance, instead of thinking of x and y as integers in the expression x+y ,the student is taught that abstraction requires thinking of x and y as objects. The term abstraction can be useful in some areas of mathematical problem solving , such as the process of building an abstract model to represent a physical problem.

The Universe shows the handworks of God and hence the existence of God. Saadia says " were we in our effort to give an account of God, to make use of only expressions which are literally true.... There would be nothing left for us to affirm except the fact of His existence."[6]

In fact, abstraction is the life's blood of mathematics and it is almost characteristic or synonymous with intelligence itself. Logic and reason in mathematics make it a useful tool in the study of science.

One of the standard mathematical activities is the proving of what is called "existence and uniqueness theorems". An existence theorem is one, which asserts that, subject to certain restrictions set down a priori, there will be a solution to such and such a problem, while a uniqueness theorem asserts that under such presuppositions a problem can have no more than one solution. "External reality is creation. And since the creation is the result of the design of a rational God .....He is the final reality, the only self-existent being"[7] .As Moses proclaimed "Hear O Israel: The Lord our God, the Lord is one" (Deut.6: 4). The designer of the universe exists and is unique. "For if He were more than one, there would apply to Him the category of Numbers and He would fall under the laws governing bodies...but the concept of quantity calls for two things neither of which can be applied to the creator"[8]. Thus, we conclude that God cannot be quantified. Yet we can talk and discuss about God and He can be the subject of meditation either through the Bible or nature. Definitely we shall have true wisdom and understanding because God will fill our thoughts.

The question has been asked "What is the basis for the way things are? What is the final reality that explains how things hold together?"[9]. An infinite-personal God is a rational God who designed the universe and things in their respective order-"a uniformity of natural causes in an open

system"[10]. He has given us the full capacity of the knowledge to explore in the different areas of studies in which we are involved.

We don't know the remote origin of writing, speech and in fact the remote origin of mathematics. However, nature has revealed sufficient expressions of patterns and symmetry, which are fundamental concepts of geometry. Thus, the question of where mathematics came from can be rested on the fact that the Designer of the universe is a Mathematician.

The mind of the creator of the universe is wonderful. The force of gravity diminishes as the second power of the distance; the planets go round the sun in ellipses. Mathematics in this view evolved precisely as a symbolic representation of the universe. It is no wonder, then that mathematics works. Paul Davies has this to say:

"The equations of physics have in them incredible simplicity, elegance and beauty, that in itself is sufficient to prove to me that there must be a God who is responsible for these laws and responsible for the universe."[11] Also, according to Alexander Polyakov "we know that nature is described by the best of all possible mathematics because God created it"[12]

John Polkinghorne says, "The rational order that science discerns is so beautiful and striking that it is natural to ask why it should be so. It could only find an explanation in a cause itself essentially rational. This would be provided by the Reason of the Creator.... we know the world also to contain beauty, moral obligation and religious experience. These also find their ground in the Creator..."[13]

The search for mathematical laws of nature was an act of devotion; it was the study of the ways and nature of God and of His plan of the universe. Galileo, Pascal, Newton and some others speak repeatedly of the harmony that God imparted to the universe through His mathematical design. The Scientists persisted in the search for mathematical laws underlying natural phenomena because they were convinced that God had incorporated these laws in His construction of the universe. Each discovery of a law of nature was hailed more as evidence of God 's brilliance than of the brilliance of the investigator.

God has designed the universe mathematically and if man has the fear of God who is the source of all knowledge can discover more mathematics.

1.3 Integration of Faith and Learning

The truths of mathematics are universal, independent, not only of individual consciousness but of social consciousness. For example, the fact that always and everywhere, regardless of time and place, politics, race or sex, 2 + 2 always equals 4 is an objective truth. One objective truth that is universal and consistent is that of salvation through the Lord Jesus.

2.1 Fractal Geometry: A New Language

Geometry is concerned in part with questions of spatial measurements. The human race has not been able to escape its fascination with geometry because the elements of geometry are imbedded in the natural

world. For instance imagine the shape and design of butterfly, starfish, petals of rose-flowers, it is amazing. In these creatures we can see both pattern and symmetry. The concept of geometry has expressed itself over and over in our daily experience. Even as people have observed pattern and symmetry in nature, they have reproduced it in their own creations. Examples are aircrafts built after the pattern of the birds of the air, robots in form of human beings, the computer with its input devices, processing unit and output devices built after the form of human systems.

In geometry there are certain basic terms such as a point, which is a location in "space" and is represented by a dot. Natural representations of points in nature include a star seen with the naked eye. A line is a collection of points that has no width and thickness but infinite length. For instance, the edge of a desk, the side of a river are physical models of lines. Another term is space which calls to mind the idea of the sky and the space surrounding this earth and indeed the totality of the physical world is a good representation of space. Space can then be thought of as the set of all points. All these are mathematical model of the things created by God.

In Gen.1:1 we read that "In the beginning God created heavens and earth.....". By implication God created circles, waves and 'fractals'. Circles are not spheres, mountains are not cones, coastlines are not circles, bark is not smooth nor does lightning travel in a straight line. Nature exhibits not simply a higher degree but an altogether different level of complexity which the classical geometry cannot model.

Fractal geometry can be used to describe many of the irregular and fragmented patterns that nature consists of mainly such as the intricate moving arrangements of the feathers on wings of a bird as it flies. The Euclidean geometry provides concise accurate description of man-made objects but it is inappropriate for natural shapes. It yields cumbersome and inaccurate descriptions. For instance, machine shops are essentially Euclidean factories: objects easily described are easily built. Furthermore, whereas Euclidean shapes are usually described by a simple algebraic formula (e.g. $x^2 + y^2 = r^2$ defines a circle of radius r), fractals, in general, are the result of a construction procedure or algorithm that is often recursive (repeated over and over) and ideally suited to computers. Thus, the computer rendering of fractals shape leaves no doubt of their relevance to nature.

2.2 Self-Similarity and Dimension

Fundamental to the understanding of fractals is the notion of *self-similarity* and *dimension* and they are closely connected.

Self-similarity is a property of a set or an object in which magnified subsets look like the whole and is identical and to each other. Examples in nature include the clouds, coastlines etc. While dimension of a set is a number that tells how densely the set occupies the metric space in which it lies and it is invariant under various stretching and squeezing of the underlying space. The exactly self-similar von Koch curve may be considered a crude model for a coastline, but it differs from the coastline in one significant aspect. Upon

magnification, segments of the coastline look like, but never exactly like, segments at different scales. In this case we have statistical self-similarity in which elements of randomness occur in the measurement of natural objects. In the measurement of the length of a coastline, the more carefully one follows the smaller wiggles the longer it becomes. A walk along a beach is longer than the drive along the corresponding coast highway.

A 1-dimensional line segment has a self similar property and can be divided into N identical parts each of which is scaled down by the ratio $r = 1/N$ from the whole.

A 2-dimensional square area can also be divided into N self similar parts each of which is scaled down by a factor $r = 1/\sqrt{N}$. This can continue to 3-dimensional, 4 dimensional etc. Thus generally, a D–dimensional self-similar object can be divided into N smaller copies of itself each of which is scaled down by a factor r such that

$$Nr^D = 1$$

Conversely, given a self-similar object of N parts scaled by a ratio r from the whole, its fractal dimension is given by

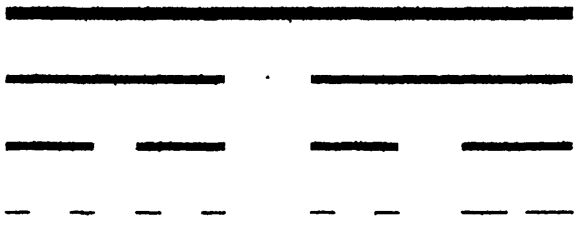
$$D = [\log N]/[\log(1/r)]$$

We can now formally define a fractal as a geometric shape which

i) is self-similar

ii) has fractional (fractal) dimension

An example is the Cantor set where each successive iteration involves removing the middle third from each line



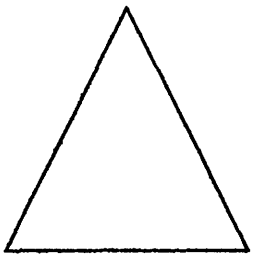
In this case N = 2,r = 1/3 and D = [log2]/[log3] = 0.6309.It fills less space than 1-dimensional object.

Illustrations

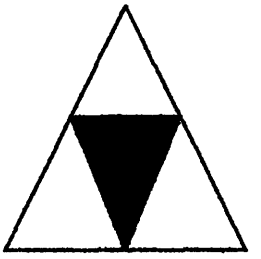
1. A train moving along railroad tracks has dimension 1.
2. A boat sailing on a lake has dimension 2
3. A plane in the sky has dimension 3
4. An unused piece of aluminum foil has dimension 2
5. When the foil is crumbled in to a ball, the dimension is 3
6. When the foil is carefully reopened, the dimension is between 2 and 3.

Below are two examples of fractals.

1. Sierpinski Triangle (Carpet)



Repeat again



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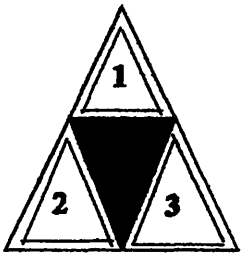

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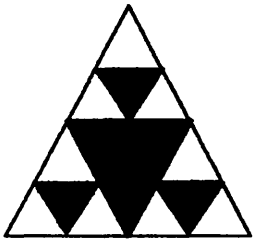

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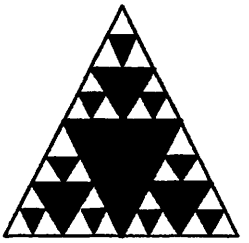
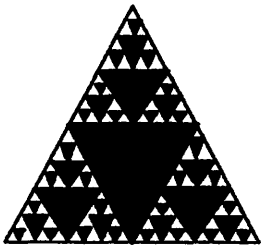


Repeat again.....



Iterate this forever

From the triangles above we note that

$$2^D = 3$$

$$\log 2^D = \log 3$$

$$D \log 2 = \log 3$$

$$D = [\log 3]/[\log 2] = 1.585 \quad \text{(not an integer)}$$

2. Von Koch - from curves to Islands

One of the most famous deterministic fractals was described by Niels von Koch and is called the Von Koch Island. This is generated as follows:

Consider starting with straight line that we will call the *initiator*

The *initiator*

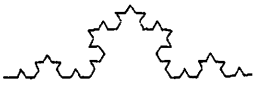
We now define a *generator*, or production rule, that states

> "Take the initiator, scale it down by a factor r=1/3, make N=4 copies and replace the initiator by these scaled down copies, oriented as shown".

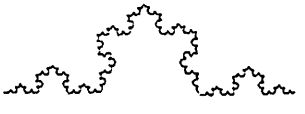
The *generator*

The *generator* is repeated...

and repeated...



and repeated...



2.3 Integration of Faith and Learning II

Self-Similarity is a property of a set in which a subset or a small part will look the same as the whole when magnified, there is a sense of consistency in this property and also there is high degree of order. In parallel, let us consider the body of Christ i.e the church. The church of God is one or the same in terms of beliefs and obedience irrespective of race, sex and nationality. Also, self-similarity shows orderliness which exhibits or displays beauty. Our God is a God of order and as such God of beauty.

The fractal dimension is invariant under squeezing and stretching of the underlying space , the children of God are in the world but not of the world, the world in which they live will be stretched and squeezed but their faith in God is to remain invariant or unmoved, to please God we need the type of Daniel's faith that would remain invariant under stress and strain.

2.4 Iterated Function Syst ems

A more unified means of generating fractals is using *iterated function systems*, a collection of contraction mappings satisfying certain conditions. In this case, fractal comes out as an ***attractor*** of the iterating function systems.

Definition. A mapping $f : X \to X$ on a metric space X is called a contraction mapping if there is a constant $0 \leq s < 1$ such that $d(f(x),f(y)) \leq s.d(x,y)$ for all x , y in X , where d is a metric on X and s a contractivity factor.

Definition. Suppose

i) K is a metric space

ii) $W = \{w_1,.....w_n\}$ is a finite set of mappings of K into itself where each w_i is a contraction.

iii) $P = \{p_i,.....p_n\}$ is a set of probabilities such that for all i $0 \leq p_i \leq 1, \sum p_i = 1$. We call the couple (W,P) an iterated function systems (ifs).

We then consider the following process: let z_o be any point in K, we randomly choose a map w_i (with probability p_i),compute $z_1 = w_i(z_o)$.We repeat this process a number of times. If all points are plotted after a sufficiently great number of iterations, they will distribute themselves approximately upon a compact set G, called the attractor of the iterated function systems. This attractor is called fractal which is uniquely associated with every iterated function sytems.

A characterization of G is given by the following theorem according to Jacques and Gagalowicz[14]

If (W,P) is an iterated function systems and G its attractor, then

$$A = \bigcup_{i=1}^{n} w_i(G)$$

One important lesson drawn from this concept is that of the ***attractor*** of the iterated function sytems, Christ says ' if I be lifted up, I will draw all men unto myself'(John...

The Holy Spirit works in the heart of men and women to attract them to Jesus Christ so that they can be like Him.

2.5 Some Applications of Fractals in Nature

As part of modern development fractals are now used in many forms to create textured landscapes and other intricate models.It is possible to create all sorts of realistic fractal forgeries. This is seen in many special effects within Hollywood movies and also in television advertisements.The 'Genesis effect' in the film "Star Trek II-The Wrath of Khan" was created using fractal landscape algorithms and in "Return of the Jedi" fractals were used to create the geography of a moon ,and to draw the outline of the dreaded "Death Star".Finally, fractal signals can also be used to model natural objects, allowing us to mathematically define our environment with a little higher accuracy than before.

According to Lamb [15], Climatologists frequently claim that patterns of climatic change are related to regular, deterministic cycles that can be linked to the motions of Solar Systems. Fractal analyses play a very important role in this study.

Finally, Australian National University Mathematicians are using fractals to study the retention of rainwater in soil and the production of improved cling wrap and sheet metal. Many other applications of fractals are

available around us in nature. All these are pointing to the fact that we have more things to explore in the nature of God, which need to make us to appreciate the wisdom of God.

2.6 Conclusion

In as much as the origin of the universe is a fact beyond the reach of direct human investigation it can be best known as the Creator Himself has revealed it. Our discussion on mathematics would not be profitable if it does not lead us to appreciate the wisdom of our creator, and the wondrous knowledge of the author of the world, who in the beginning created the world out of nothing and set everything in number, measure and weight, and then in time and age of man formulated mathematics which reveals fresh wonders the more we study it.

Finally, since many "natural processes may be observed, manipulated, and analyzed, they are subject to learning by discovery and because nature exhibits purposeful design, nature study is one avenue toward a knowledge of its designer"[16]

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